Function Spaces

Mid-Semester Exam

September 2025

Maximum marks: 30 Duration: 2 hours and 30 minutes

1. Let K be a compact subset of a metric space X and $x \in X \setminus K$. Show that there exists an open set U containing K and an open set V containing x for which $U \cap V = \emptyset$. (3 marks)

2. Justify your answer for the following:
   (i) For any uncountable set X, is there always a metric on X with respect to which X is not separable? (1 mark)
   (ii) Let X, Y be metric spaces and $f : X \to Y$ be a function. Is it true that if Graph(f) is closed, then f is continuous?
   [Note that: Graph(f):= $\{(x, f(x)) : x \in X\} \subseteq X \times Y$] (3 marks)

3. (i) State Lebesgue's covering lemma. (2 marks)
   (ii) Show by an example that Lebesgue's covering lemma is false when X is not a compact metric space. (3 marks)

4. Justify your answer for the following:
   (i) Let $\{f_n\}$ be a sequence of bounded real-valued functions on a metric space X that converges uniformly to a function f on X. Is f bounded? (3 marks)
   (ii) Does the sequence of functions $\{f_n\}_{n=1}^{\infty}$, where $f_n : (0, 1) \to \mathbb{R}$ defined by
   $$f_n(x) = \frac{n}{nx + 1}, \quad x \in (0, 1)$$
   converge uniformly on $(0, 1)$? (2 marks)

5. Show that the function $f : [0, \infty) \to \mathbb{R}$ defined by
   $$f(x) = \ln(1 + e^{-x}), \quad x \in [0, \infty)$$
   has a unique fixed point. (3 marks)

6. Let X be a complete metric space and $T : X \to X$ be a map. Suppose there exists $n \in \mathbb{N}$ such that $T^n := T \circ T \circ \cdots \circ T$ (n iterates) is a contaction on X. Show that T has a unique fixed point in X. (3 marks)

7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $C[a, b]$ with no uniformly convergent subsequence. For $n \in \mathbb{N}$, define a function F_n by
   $$F_n(x) = \int_a^x \sin(f_n(t))dt, \quad x \in [a, b].$$
   Does the sequence $\{F_n\}_{n=1}^{\infty}$ have a uniformly convergent subsequence? (Justify) (3 marks)

8. (i) State Stone-Weierstrass Theorem. (2 marks)

   (ii) Show that for a compact set $K \subseteq \mathbb{R}^n$, $C(K)$ is separable. (3 marks)

9. Give an example of a metric space X and an algebra $\mathcal{A} \subseteq C(X)$ such that $\mathcal{A}$ contains more than one element, fails to separate points and vanishes at some point. (3 marks)